Predicting the values of Archie 'a' , 'm' parameters from NMR data using a fractal model of pore space

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Summary

In this paper, the electric conductivity of a brine-saturated rock is modeled, visualizing the pore space of a rock, as a bundle of brine-filled capillaries. Further, in this visualization of the pore space, the capillary number density and the tortuous length of a given capillary segment, exhibit fractal behavior, making pore space a bi-fractal object, characterized by two fractal dimensions. The forward model of rock-conductivity at 100% brine-saturation,based on the above visualization, has the rock porosity, and the values of the fractal dimensions of the pore space respectively, as the model inputs. It is found from the modeling that the forward model of rock conductivity, predicts Archie behavior of rock conductivity with respect to the rock porosity and the conductivity of the brine, saturating the rock, with the Archie 'a' and 'm' parameters being functions of the fractal dimensions of pore space. A method of computing the values of the fractal dimensions of the pore space of rocks, from the NMR T_2-distribution of the rock, is also presented in this paper, for completeness.

Introduction

The electric conduction through a porous clean rock whose grains are non-conductive electrically, and which is saturated with an electrolyte, is effected, through the motion of the ions of the electrolyte within the pore space of the rock. So much so, the gross conductivity of the rock would depend mainly upon the geometry of the pore space of the rock and the conductivity of the electrolyte. The principal theme of the analysis presented in the first part of this paper concerns an investigation into the relationship between the gross conductivity of the rock and the conductivity of the electrolyte saturating its pore space, assuming a fractal model of the pore space of the rock. The analysis presented in the second part of the paper concerns a method of evaluating the Archie parameters 'a', 'm', based on the results of the abovementioned analysis, from NMR log data.

Theory and/or Method

Model of the pore space of a rock, used in this study

For the purpose of modeling, the geometry flow of ions within the pore fluid in response to an electric potential difference set up between two parallel planes within the rock is considered as having the geometry of a bundle of non-intersecting capillaries. In the pore space-model, used, the border of the cross section of a capillary is assumed to be a fractal, which approximates to a circle, at the length scales comparable to the cross-sectional size of the capillary. Further, as per the model, the value of the fractal dimension of the border of the cross section of a capillary is the same as the value of the fractal dimension of the border of the cross section of any other capillary amongst the set of capillaries that comprise together, the pore space.

The electrical conductance of a volume of rock is modeled as the gross electric conductance of the bundle of non-intersecting capillaries referred to, above, with each capillary being a conductor by virtue of it having been saturated with electrically conducting brine. For this reason, in this paper, the term 'capillary' is used interchangeably with the term 'conductor'. While the cross section of a capillary is assumed to be uniform throughout its length, the cross sectional size of capillaries however, can vary from capillary to capillary and has a distribution over their number-density, in the model. The capillaries are assumed to be tortuous in general. The axis of a capillary is assumed to be a fractal, and therefore, that the distribution of the tortuous length of a capillary over the capillary size follows a fractal law, according to which, the tortuous length l_t of a capillary passing through two points

A, B scales over its diameter of cross section, denoted as $'D'$ as

$$l_t = l_0^{D_L} G^{1-D_L} \quad (1)$$

It can be noted that Eq. (1) has the form of Richardson's equation for the coastline of Britain as cited at, Mandelbrot, 1982, and which is given below.

$$L(G) = MG^{1-D_R} \quad (1a)$$

Here, $'G'$ denotes the length measure used in assessing $L(G)$, the length of the coastline between any two given points lying on it and D_R is recognized as the dimension of the fractal which is, the segment of the coast line of Britain between the two given points lying on it, mentioned as above. The more convoluted the coast line between the two points mentioned above, the more the value of the fractal dimension. The value of unity for the fractal dimension corresponds to a straight coastline, which implies that the value of M is the length of a straight line-segment whose ends are the two points mentioned above. The value of D_R cannot exceed 2.0 because for the value of D_R equal to 2.0 the coastline would become a space-filling curve. Thus, $1 \leq D_R \leq 2$.

Considering the axial line of a capillary segment as a fractal, the length of a capillary segment whose ends are points A, B (say), would be given as stated at Eq. (1), where l_t denotes the tortuous length of the capillary segment whose extremities are A and B, respectively and $l_0 = AB$. In the context of the capillary bundle model of the pore space alluded to, in the foregoing discussion, G can be interpreted as the characteristic length scale of the sub-space of the pore space that a capillary would represent.

The characteristic length scale of the capillary subspace represented by a capillary, of the pore space, equals the diameter D of the capillary, and is the most realistic choice for G in equation (1). The resulting equation which gives, how l_t would scale on D is:

$$l_t = l_0^{D_L} D^{1-D_L} \quad (2)$$

Eq. (2) is the fractal scaling relationship developed by Wheatcraft and Tyler (Wheatcraft and Tyler, 1988) given below:

$$l_t = l_0^{D_L} \varepsilon^{1-D_L} \quad (2a)$$

with, the characteristic length scale (ε) of the subspace of the pore space, represented by a given capillary, being considered, as value of the diameter, of the given capillary.

In light of the range of D_R mentioned above, and, since D_L has to have a similar range of values, we have, $1 \leq D_L \leq 2$.

If D_{max} stands for the maximum capillary diameter, the number of capillaries $n(> D)$ which cross unit area of a plane transverse to them and whose diameter is $> D$,is assumed to scale over$D$ the diameter of a capillary as,

$$n(\geq D) = D_{max}^{D_F} D^{-D_F} \quad (3)$$

The reasoning behind equation (3) is as below:

The section of the set of capillaries, on a transverse-plane, would be a set of fractals. When a set of islands or a set of lakes at the mean sea level, and situated within a certain area is considered, the distribution of the area of an island or that of a lake, is known to exhibit fractal behavior, characterized as:

$$N(\geq a) = \left(\frac{a_{max}}{a}\right)^{-\frac{\Delta}{2}} \quad (4a)$$

where,$'\Delta'$ denotes the area fractal dimension applicable for the particular set, respectively (Mandelbrot, 1982). The value of Δ and the value of the fractal dimension of the border of the object (island / lake) will be the same. Further, the value of Δ is constrained to lie between a value of $\Delta = 1$, (a collection of smooth shore-lines), and $\Delta = 2$, (a collection of shore-lines so irregular, that they would be space-filling), (Seekell, et al., (2013)). The fractal model discussed above, has been extended to the case of collections of contact defects which show up on engineering surfaces, with the borders of the contact defects assumed to be fractals (Majumdar and Bhushan (1990)). The fractal scaling similar to Eq. (4a), which results, has been successfully used by them (Majumdar and Bhushan (1990)) to model the cumulated distribution of the area of a contact defect amongst a set of contact defects that show up on engineering surfaces. In this context, the symbol $'a_{max}'$ would now signify the maximum value that a contact defect would have, and the symbol 'a' would signify the area of a contact defect.

The fractal model which leads to Eq. 4a. has been extended in this paper, to the set of capillary-cross-

section border fractals, with the 'area' denoted by the symbol $'a'$, now signifying the area of cross section of a capillary, and the symbol D_F denoting the area fractal dimension relevant to the collection of the capillary-cross sections on a transverse plane. Note that the implicit assumption is, (as has been for the case of sets of islands or lakes at the mean sea level or a set of contact defects showing up on an engineering surface), that the same fractal dimension characterizes the set of capillary-cross-section border curves, for a cross-section of the capillaries-set on a transverse plane.

Consequently, the cumulated distribution of the number of capillaries intersected by a unit area lying on a plane transverse to the capillaries, is assumed to scale over the cross sectional area of a capillary as per the fractal distribution indicated above at (4a), with the symbol $'D_F'$.replacing the symbol $'\Delta'$, and the symbols $'a_{max}','a'$ denoting the maximum value of the area of cross-section of a capillary, and the area of cross section of a given capillary, respectively.

Since, the cross-sectional area of a capillary would vary as the square of its size of the cross section, the above would imply that the number of capillaries $n(>D)$ which cross unit area of a plane perpendicular to them and whose equivalent diameter is $>D$ would scale over D as:

$$n(\geq D) = \left(\frac{D_{max}}{D}\right)^{-D_F} \quad (4b)$$

Eq. 4b can be rearranged to have the form of Eq. 3 above. The value of D_F the area fractal dimension equals the value of the fractal dimension of the border enclosing the area of cross-section of a capillary. This being the case, the value of D_F is constrained as $1 \leq D_F \leq 2$

From Eq. 4b, or, equivalently, Eq. 3 we have,

If $'n(D)'$ denotes the number of capillaries whose diameter lies within the interval $(D, D+dD)$,

$$n(D) = n(\geq D) - n(\geq [D+dD]) = D_F D_{max}^{D_F} D^{-(D_F+1)} \quad (4)$$

While it is a necessary condition that $\left(\frac{D_{min}}{D_{max}}\right) \cong 0$, for a fractal model to be applicable to model the pore space of a porous medium (see Appendix 1), it is also a known fact that the inequality $\left(\frac{D_{min}}{D_{max}}\right) < 0.01$ holds for the case of the pore space of most sedimentary rocks.

We explicitly state here, that it is a model assumption that, $\left(\frac{D_{min}}{D_{max}}\right) < 0.01$.

Further, the value of D_F cannot exceed 2.0, and the value of D_F cannot be less than 1.0 (see also, Yu and Cheng (2002)).

The forward model of rock conductivity

Consider a unit cube of rock. Let a potential difference of ΔV be set up across the pair of opposite faces of the cube. In accordance with our model, the ionic flow paths form flow tubes, each flow tube being coincident with a capillary, of the capillary bundle which is supposed to model the pore space insofar as the flow of ions or the hydraulic flow of the pore-filling fluid between any two planes cutting the rock, in response to a pressure differential set up between the planes, is considered. Let I denote the current.

Let C_r and σ respectively represent the conductance of unit cube of rock and the bulk conductivity of the rock, respectively.

$$I = C_r \Delta V \quad (5)$$

The contribution I_t of a given capillary to the current flow is:

$$I_t = \Delta V C_D = \Delta V \sigma_F \frac{\pi D^2}{4 l_t} = \Delta V \sigma_F \left(\frac{\pi}{4}\right) \frac{D^2}{l_0^{D_L} D^{1-D_L}} = \Delta V \sigma_F \left(\frac{\pi}{4}\right) \frac{D^2}{D^{1-D_L}} \quad (6)$$

Where, we have used Eq.2 to relate I_t, I_0 and D, and the fact that, $l_0 = 1$, in the present context. In Eq. 6, the symbol $'C_D'$ denotes the conductance of a single capillary.

The total current I is given, in light of Eq. 6, by:

$$I = \int_{D_{min}}^{D_{max}} \Delta V \sigma_F \left(\frac{\pi}{4}\right) \frac{D^2}{D^{1-D_L}} n(D) dD = \quad (7)$$

Substituting from Eq. 4 into Eq. 7, carrying out the integration, and simplifying:

$$I = \Delta V \sigma_F \left(\frac{\pi}{4}\right) D_F D_{max}^{D_F} \frac{1}{(D_L - D_F + 1)} \left[D_{max}^{(D_L - D_F + 1)} - D_{min}^{(D_L - D_F + 1)}\right]$$

which can be further simplified without loss in accuracy to:

$$I = \Delta V \sigma_F \left(\frac{\pi}{4}\right) D_F D_{max}^{D_L + 1} \frac{1}{(D_L - D_F + 1)} \quad (8a)$$

While arriving at Eq. 8a, we have used the fact that, in our model $\left(\frac{D_{max}}{D_{min}}\right) > 100$ is a model assumption, rendering little loss of accuracy in ignoring $D_{min}^{(D_L - D_F + 1)}$ in comparison with $D_{max}^{(D_L - D_F + 1)}$, given that $1 \leq D_L \leq 2$ and $1 \leq D_F \leq 2$ as discussed above, and therefore, $(D_L - D_F + 1)$is always ≥ 0.

Substituting for I from Eq. 5 into Eq. 8a and cancelling out ΔV throughout, we get,

$$C_r = \sigma_F \left(\frac{\pi}{4}\right) D_F D_{max}^{D_L + 1} \frac{1}{(D_L - D_F + 1)} \quad (8)$$

Porosity of the rock

The porosity of the rock as per our model is computed as,

$$\varphi = \int_{D_{min}}^{D_{max}} \frac{\pi D^2}{4 l_t} l_t n(D) dD \quad (9)$$

Substituting from Eq. 4 into Eq. 9, carrying out the integration, and, since the inequality $\left(\frac{D_{max}}{D_{min}}\right) > 100$ is a model assumption, and the values of the fractal dimensions D_L, D_F individually range between 1.0 and 2.0, the inequality $(D_L - D_F + 1) \geq 0$ always holds given the abovementioned ranges of the values of the fractal dimensions. Consequently, the equation for φ can be stated as,

$$\varphi = \left(\frac{\pi}{4}\right) D_F D_{max}^{3 - D_L} \frac{1}{(3 - D_L - D_F)} \quad (10)$$

We note that since C_r happens to be the conductance of a unit cube, its magnitude is equal to σ. Hence,

$$\sigma = \sigma_F \left(\frac{\pi}{4}\right) D_F D_{max}^{D_L + 1} \frac{1}{(D_L - D_F + 1)} \quad (11)$$

Eq. 10 and Eq. 11 together imply,

$$\sigma = \sigma_F \left\{ \left(\frac{\pi}{4}\right)^{1 - \frac{(D_L + 1)}{(3 - D_L)}} D_F^{\,1 - \frac{(D_L + 1)}{(3 - D_L)}} \left(\frac{3 - D_L - D_F}{D_L - D_F + 1}\right)^{\frac{D_L + 1}{3 - D_L}} \right\} \varphi^{\frac{D_L + 1}{3 - D_L}} \quad (12)$$

Eq. 12 is the forward model of rock conductivity as per our model.

Note that the term $\left(\frac{\pi}{4}\right)$ in the forward model for σ is a parameter related to the geometry of the cross section of the capillary,(which in turn, is an expression of the general pore cross section), and can in general, be replaced by a model input parameter G, if one wishes to make the model more flexible.

We note that the fractal dimensions of pore space, as per the model assumed, are independent of the porosity. Therefore Eq. 12 has the Archie form of the equation connecting porosity, pore fluid conductivity and rock conductivity at 100% pore fluid saturation within the rock. Inspecting Eq. 12, it can be noted that the Archie 'a' and Archie 'm' values are predicted by the model to be as below:

$$a = \left(\frac{\pi D_F}{4}\right)^{1 - \frac{(D_L + 1)}{(3 - D_L)}} \left(\frac{3 - D_L - D_F}{D_L - D_F + 1}\right)^{\frac{D_L + 1}{3 - D_L}} \quad (13)$$

$$m = \frac{D_L + 1}{3 - D_L} \quad (14)$$

Evaluation of the fractal dimensions D_L and D_F using NMR log data

At the outset we note that the value of porosity is a standard output of NMR logs.

Consider the NMR T_2 distribution of a unit cube of the rock. This distribution is visualized as a histogram comprising M classes. Let $\psi(T_2, T_2 + \Delta T_2)$ denote the pore volume per unit rock volume, which corresponds to those pores whose NMR T_2 lies within the interval, $(T_2, T_2 + \Delta T_2)$. Let ρ denote the grain surface relaxivity and let ΔD denote the spread in the diameter of the pores which pertain to $(T_2, T_2 + \Delta T_2)$.

$$D = 4\rho T_2 \quad (15)$$

$$\Delta D = 4\rho \Delta T_2 \quad (16)$$

$$\psi(T_2, T_2 + \Delta T_2) = V_p^{\Delta D}$$

$$V_p^{\Delta D} = n(D) \frac{\pi (D(T_2))^2}{4} l_t (D(T_2) \Delta D \quad (17)$$

$$V_p^{\Delta D} = D_F D_{max}^{D_F} D^{-(D_F+1)} \frac{\pi(D(T_2))^2}{4} D^{1-D_L} \Delta D$$

(17), where Eq. 2 has been used for l_t with unit value for l_0

Substituting for ΔD from Eq. 16 into Eq. 17, simplifying and rearranging, we get,

$$\frac{\psi(T_2, T_2+\Delta T_2)}{\rho \Delta T_2} = \pi(D(T_2) n(D(T_2) l_t(D(T_2)$$

(18)

Substituting for $n(D), l_t(D)$ from Eq. 4 and Eq. 2 respectively into Eq. 18 and simplifying, we get, (as the distance between a pair of opposite faces of a unit cube is 1.0 making the straight line length between the ends of a capillary equal to 1.0),

$$\frac{\psi(T_2, T_2+\Delta T_2)}{\rho \Delta T_2} = \pi D_F D_{max}^{D_F} (D(T_2)^{2-D_F-D_L}$$

(19)

Indexing the equation similar to Eq. 19, that would be relevant for the j^{th} histogram-class with the subscript *'j'*, we get M number of equations,

$$\frac{\psi(T_{2j}, T_{2j}+\Delta T_{2j})}{\rho \Delta T_{2j}} = \pi D_F D_{max}^{D_F} (D(T_{2j})^{2-D_F-D_L}$$

For, $j = 1 \text{ to } M$ (20)

Taking natural logarithms on both sides and using Eq. 15 we get,

$$\ln\left[\frac{\psi(T_{2j}, T_{2j}+\Delta T_{2j})}{\rho \Delta T_{2j}}\right] = \ln\left[\pi D_F D_{max}^{D_F}\right] + (2 - D_F - D_L)\left[\ln(4\rho T_{2j})\right]$$

for, $j = 1 \text{ to } M$ (21)

Denoting the LHS of Eq. 21 as 'y_j', $\ln\left[\pi D_F D_{max}^{D_F}\right]$ as 'c', $(2 - D_F - D_L)$ as'λ' and , denoting $\left[\ln(4\rho T_{2j})\right]$ as x_j, the set of equations (21) are stated as

$$y_j = c + \lambda x_j \text{ for } j = 1 \text{ to } M \quad (22)$$

The values of c and λ for each depth level are obtained by performing a least square fit.

We now have,

$$e^c = \left[\pi D_F D_{max}^{D_F}\right] \quad (23)$$

The value of D_{max} for a given depth level is known since $D_{max} = T_{2max}$ which is known as it can be evaluated as the logarithmic mean of the left and right end-points of the histogram class for which the cumulated pore volume is 95% of the total pore volume for the given T_2 distribution. The value of the LHS of Eq. 23 is known at this point as discussed above.

Eq. 23 is now be inverted for the value of D_F as follows:

Starting from a value of 1.0 for, D_F, the RHS of Eq. 23 is evaluated till the value of the RHS exceeds that of e^c by a number nearest to and which exceeds δ where δ is a pre-set small positive number.

Let $D_{Fexceed}$ denote the value of $\left[\pi D_F D_{max}^{D_F}\right]$ when this happens. Then, the value of D_F is given as,

$$D_F = D_{Fexceed} - \delta/2 \quad (24)$$

Now, the value of λ, which has also been obtained from the least square fit referred to above, is nothing but the value of $(2 - D_F - D_L)$. Thus,

$$\lambda = (2 - D_F - D_L) \quad (25)$$

Substitution of the values of λ and of D_F already obtained, into Eq. 25 and solving for D_L, yields the value of D_L.

Discussion

The border of the cross section of a capillary is a fractal in the model. However, for the purposes of calculating the conventional current flow, in response to an electric potential gradient, it has been assumed here that that the current flow would be the same if each capillary had its cross section as circular and of a size such that the area of cross section enclosed by this circle equals the area of cross section of the original capillary. What the symbol D used in this treatment, stands for is the size if the circular cross section defined as above. This is consistent with the assumption, that a capillary cross section mimics a circle at scales of the order of or higher than the cross-sectional size of the capillary.

The length of a capillary varies with the cross-sectional size of the capillary. Hence, we do not use the assumption that the areal porosity of a transverse section of the capillary bundle approximates to the porosity of the medium.

Further, in our model, the pore space of a porous medium is a bi-fractal. Hence the fractal dimension D_F of our model is not conceptually, the same as the fractal dimension D_f of for example Yu, Li., 2001, who give a relation between the maximum pore dimension minimum pore dimension, dimension of the Euclidian space in which the fractal pore space

(characterized by a single fractal dimension D_f and the porosity, as $\varphi = \left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{D_E - D_f}$ where, D_E is the dimension of the Euclidian space in which the fractal pore space and λ denotes pore size with the subscripts standing for the respective end points of the pore size.

The work flow assumes that the fractal model of pore space is valid for the porous medium. The necessary condition for the validity of fractal model of pore space is given at Appendix 1. While this condition is satisfied by the pore space of most sedimentary rocks, it is advisable to confirm that the condition is met with prior to applying the work flow.

The work flow assumes implicitly that the pores are largely interconnected. This means that electrically isolated pores from their neighbors are absent or form a small fraction of the pore assemblage.

The work flow assumes that bulk relaxation effects on the NMR T_2 distribution are minimal. This means that cracks, large pores (mega pores) such and dissolution pores and fractures are absent / have minimal presence so that the t2 distribution is largely controlled by surface relaxation. Thus, the methodology presented in this paper is best suited for granular rocks free of dissolution pores, cracks, large solution channels and fractures and so on.

Conclusions

A viable work flow for the prediction of Archie 'a' and 'm' parameters using NMR data has been demonstrated, that is consistent and feasible to apply for the rock types discussed above.

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Appendix 1

Necessary condition for a fractal analysis of the pore space of a porous medium to be applicable

From Eq. 4 in the body of this paper, the probability density $P(D)$ of the capillary size is given by,

$$P(D) = \frac{n(D)}{N} = \frac{D_F D_{max}^{D_F} D^{-(D_F+1)}}{N} \quad \text{(A.1)}$$

where N denotes the total number of capillaries

$$N = D_F D_{max}^{D_F} D_{min}^{-(D_F+1)} \quad \text{(A.2)}$$

From Eq. A1 and Eq. A2 we have,

$$P(D) = D_{min}^{D_F} D^{-(D_F+1)} \quad \text{(A.3)}$$

Since $P(D)$ is a probability density function, necessarily,

$$\int_{D_{min}}^{D_{max}} P(D)dD = 1 \quad \text{(A.4)}$$

Eq. A4 implies

$\int_{D_{min}}^{D_{max}} D_{min}^{D_F} D^{-(D_F+1)} dD = 1$ necessarily, which in turn implies,

$\left(\frac{D_{min}}{D_{max}}\right)^{D_F} = 0$ because, otherwise, Eq. A4 would not hold true.

In real world the above condition can be interpreted as the condition that

$\left(\frac{D_{min}}{D_{max}}\right)^{D_F} \ll 1$ must be satisfied, for a porous medium to be considered as a medium whose pore space can be modeled as a fractal.

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